APPENDIX A
PARTIAL FRACTIONS
Proof of Theorem 4 in Chapter 3

Theorem 4: If $f(x) = \dfrac{a(x)}{b(x)}$ where $a(x)$ and $b(x)$ are real polynomials then $f(x)$ is a real polynomial plus a sum of rational functions of the form $\dfrac{k}{(x-b)^n}$ and $\dfrac{kx+h}{(x^2+ex+f)^n}$ where each n is a positive integer, the coefficients are real, and the quadratic denominators have non-real zeros.

Proof: Let $b(x) = p_1(x)^{n_1} p_2(x)^{n_2} \dots p_r(x)^{n_r}$ where each $p_i(x)$ is a real factor and is either linear or a quadratic that cannot be factorised over $\mathbb{R}$.

For each i let $Q_i(x) = \dfrac{b(x)}{p_i(x)^{n_i}}$.

Since the $Q_i(x)$ are coprime we can write

$1 = Q_1(x)D_1(x) + \dots + Q_r(x)D_r(x)$ for some real polynomials $D_i(x), \dots, D_r(x)$.

Hence $a(x) = Q_1(x)D_1(x)a(x) + \dots + Q_r(x)D_r(x)a(x)$.

So $\dfrac{a(x)}{b(x)} = \dfrac{b(x)D_1(x)a(x)}{p_1(x)^{n_1}} + ... + \dfrac{b(x)D_r(x)a(x)}{p_r(x)^{n_r}}$.

Each of these terms is a polynomial plus a term of the form $\dfrac{s(x)}{(x-b)^n}$ where deg $s(x) < n$ or $\dfrac{t(x)}{(x^2+ex+f)^n}$ where $e^2 < 4f$ and deg $t(x) < 2n$.

It remains to show that $\dfrac{s(x)}{(x-b)^n}$ can be expressed as a sum of terms of the form $\dfrac{k}{(x-b)^m}$ where $1 \leq m \leq n$ and $\dfrac{t(x)}{(x^2+ex+f)^n}$ can be expressed as a sum of terms of the form $\dfrac{kx+h}{(x^2+ex+f)^m}$ where $1 \leq m \leq n$.

$\mathbf{\dfrac{s(x)}{(x-b)^n}}$ **:** We prove the result by induction on n.

It is clearly true for $n = 1$.

Suppose it is true for n.

Let $s(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$.

Then $s(x) = a_n(x-b)^n + b_{n-1}x^{n-1} + ... + b_1 x + b_0$ for some real numbers b_i.

So $\dfrac{s(x)}{(x-b)^{n+1}} = \dfrac{a_n}{x-b} + \dfrac{b_{n-1}x^{n-1} + ... + b_1x + b_0}{(x-b)^n} \cdot \dfrac{1}{(x-b)}$.

By induction $\dfrac{b_{n-1}x^{n-1} + ... + b_1x + b_0}{(x-b)^n}$

$= \dfrac{A_1}{(x-b)} + \dfrac{A_2}{(x-b)^2} + ... + \dfrac{A_n}{(x-b)^n}$ for some real numbers A_i.

Hence $\dfrac{s(x)}{(x-b)^{n+1}} = \dfrac{a_n}{x-b} + \dfrac{A_1}{(x-b)^2} + ... + \dfrac{A_n}{(x-b)^{n+1}}$

and so the result holds for $n + 1$.

$\mathbf{\dfrac{t(x)}{(x^2 + ex + f)^n}}$ **:** We prove the result by induction on n.

It is clearly true for $n = 1$.

Suppose it is true for n.

Let $t(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + ... + a_1x + a_0$.

Then $t(x) = (a_{2n+1}x + (a_{2n} - 2a_{2n+1}e))(x^2 + ex + f)^n +$ $b_{2n-1}x^{2n-1} + ... + b_1x + b_0$ for some real numbers b_i. (Just check that the coefficients of x^{2n+1} and x^{2n} are the same.)

So $\dfrac{t(x)}{(x^2 + ex + f)^{n+1}} = \dfrac{a_{2n+1}x + (a_{2n} - 2a_{2n+1}e)}{x^2 + ex + f}$

$$+ \frac{b_{2n-1}x^{2n-1} + \ldots + b_1 x + b_0}{(x^2 + ex + f)^n} \cdot \frac{1}{(x^2 + ex + f)} .$$

By induction $\dfrac{b_{2n-1}x^{2n-1} + \ldots + b_1 x + b_0}{(x^2 + ex + f)^n} = \dfrac{A_1 x + B_1}{(x^2 + ex + f)} +$

$\ldots + \dfrac{A_n x + B_n}{(x^2 + ex + f)^n}$ for some real numbers A_i and B_i.

Hence $\dfrac{t(x)}{(x^2 + ex + f)^{n+1}}$

$$= \frac{a_{2n+1}x + (a_{2n} - 2a_{2n+1})e}{x^2 + ex + f} + \frac{A_1 x + B_1}{(x^2 + ex + f)^2}$$

$$+ \ldots + \frac{A_n x + B_n}{(x^2 + ex + f)^{n+1}}$$

and so the result holds for $n + 1$.